Heap Sort

Lecture 02.07

by Marina Barsky

We can sort using Heaps!

HeapSort (array A of size n)

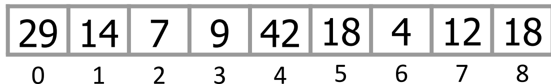
```
create an empty max-heap H

for i from 0 to n-1:
    H.insert(A[i])
for i from n-1 downto 0:
    A[i] ← H.pop()
```

After Array elements inserted into a Heap: sort using pop(*extract max*)

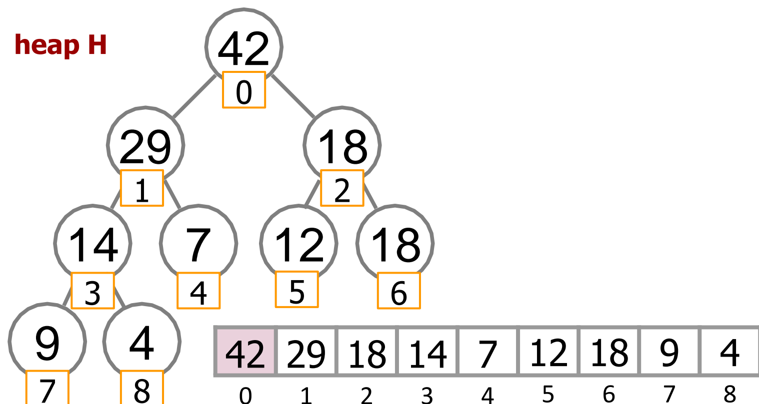
heap H

42	29	18	14	7	12	18	9	4
0	1	2	3	4	5	6	7	8

input array A



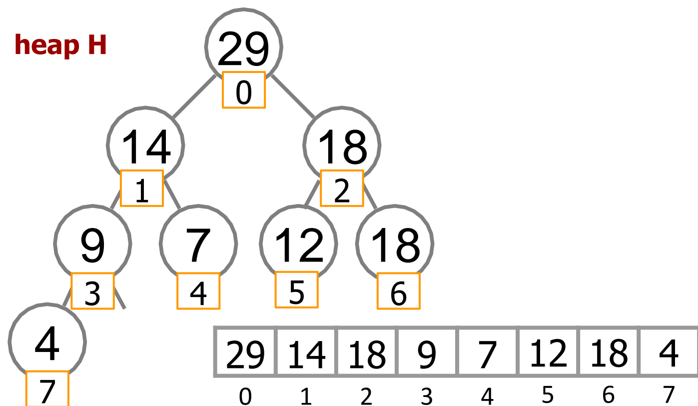
After Array elements inserted into a Heap: *extract max*



42	29	18	14	7	12	18	9	4
0	1	2	3	4	5	6	7	8

29	14	7	9	42	18	4	12	42
0	1	2	3	4	5	6	7	8

After Array elements inserted into a Heap: *extract max*

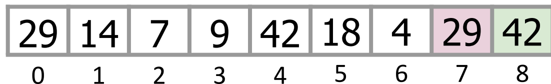


After Array elements inserted into a Heap: *extract max*

heap H

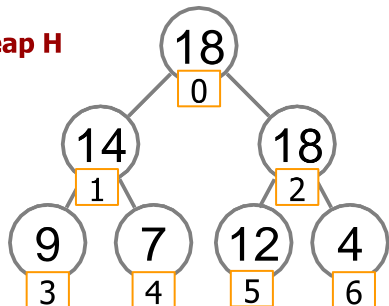
29	14	18	9	7	12	18	4
0	1	2	3	4	5	6	7

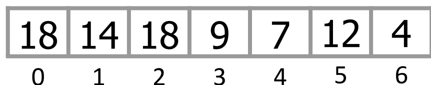
input array A



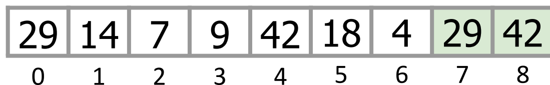
After Array elements inserted into a Heap: *extract max*

heap H

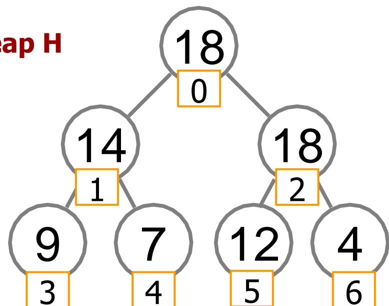


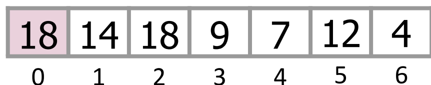


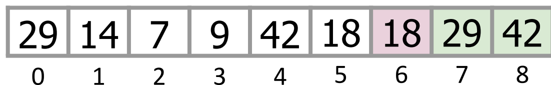
input array A



After Array elements inserted into a Heap: *extract max*







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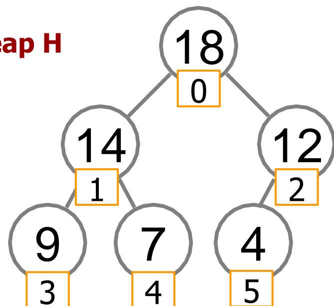
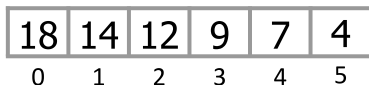
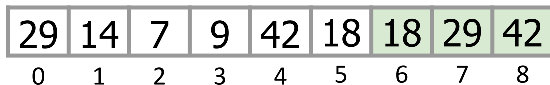
heap H

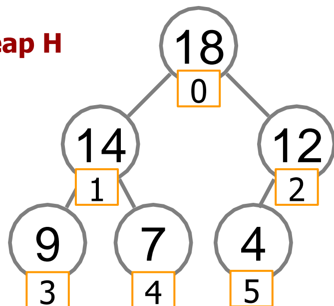


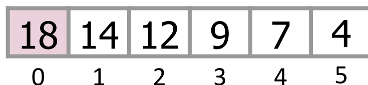


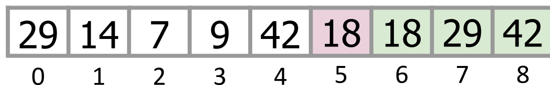
input array A



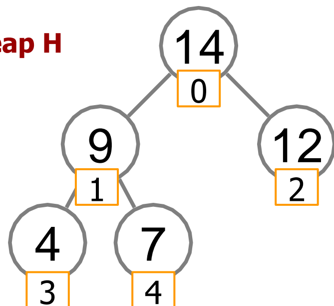
After Array elements inserted into a Heap: *extract max*

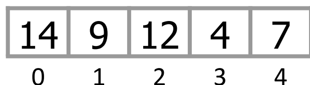


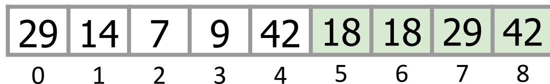




After Array elements inserted into a Heap: *extract max*

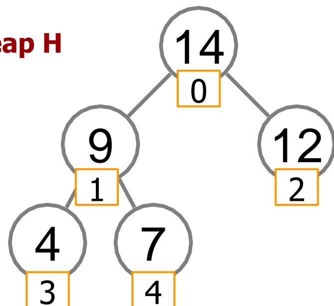


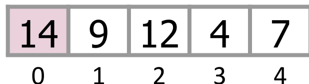




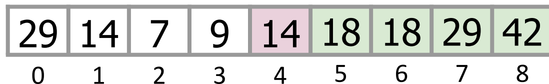
After Array elements inserted into a Heap: *extract max*

heap H

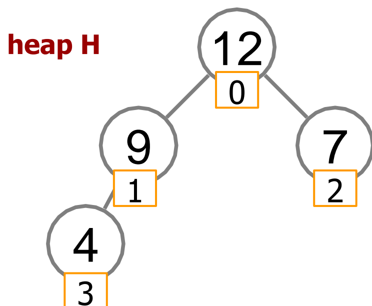




input array A



After Array elements inserted into a Heap: *extract max*



We continue extracting max and rearranging the remaining heap elements until H becomes empty and A is filled with sorted values...

12	9	7	4
0	1	2	3

sorted portion of A

input array A

29	14	7	9	14	18	18	29	42
0	1	2	3	4	5	6	7	8

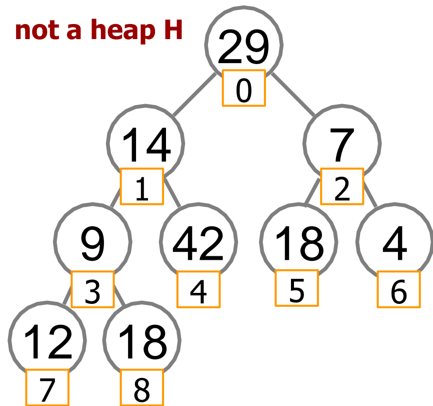
Heapsort: naive

- The resulting algorithm is comparison-based and has running time $O(n \log n)$ (asymptotically optimal)
- Natural generalization of *selection sort*: instead of simply scanning the rest of the array to find the maximum value, use a smart data structure
- Not in-place: uses additional space $O(n)$ to store the heap

In-place Heapsort: all is done inside the input array

- Turn input array A into a heap by rearranging its elements
- After this, extract max at A[0] and swap it with the element A[size-1]
- Decrement heap size
- Restore heap
- Continue until heap size=1

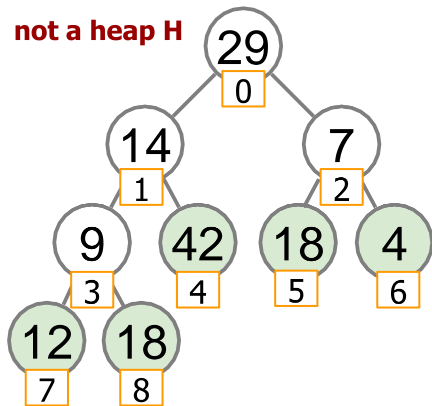
How to Heapify an array



➔ Lets' go bottom up and repair heap property for all subtrees rooted at current node

29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

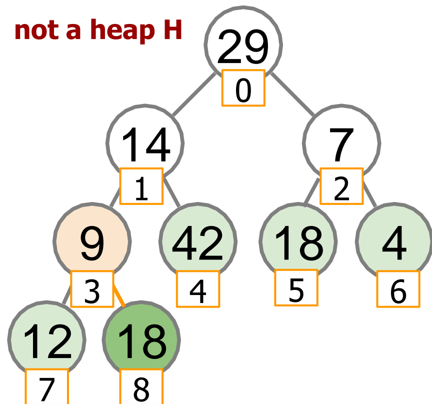
How to Heapify an array



29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

- ➔ Lets' go bottom up and repair heap property for all subtrees rooted at current node
- ➔ If current node is a leaf, then it does not need to be repaired
- ➔ How do we find the first from the end node that is not a leaf?

How to Heapify an array



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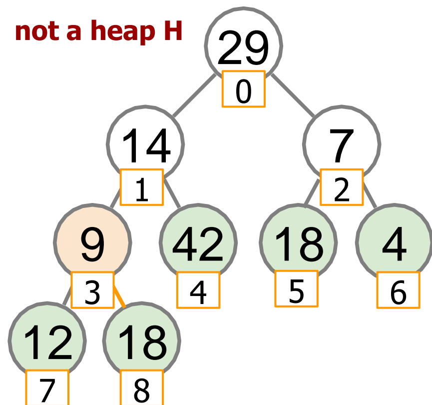
29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

We find the parent of the last leaf H[size -1]:

parent(H[i]) = $H[\lfloor (i-1)/2 \rfloor]$

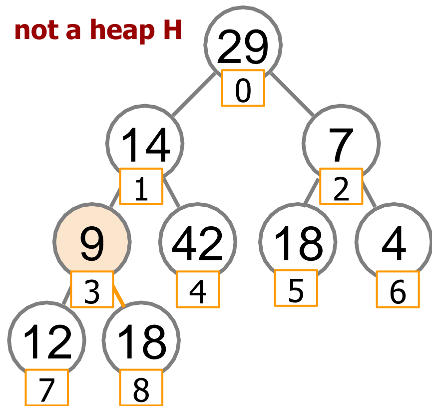
How to Heapify an array



➔ We need to go from element at position $i = \lfloor (8-1)/2 \rfloor = 3$ until position 0 and repair heap violations by calling sift_down(i)

29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

How to Heapify an array

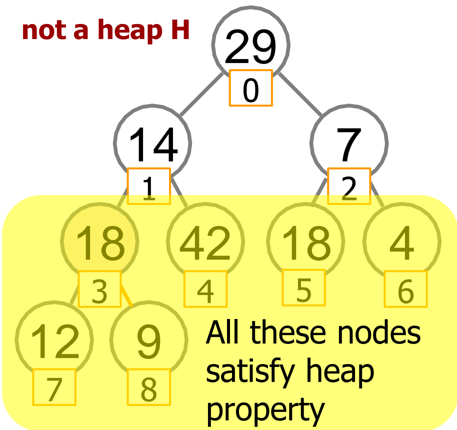


➔ We need to go from element at position $i = \lfloor (8-1)/2 \rfloor = 3$ until position 0 and repair heap violations by calling sift_down(i)

sift_down(3)

29	14	7	9	42	18	4	12	18
0	1	2	3	4	5	6	7	8

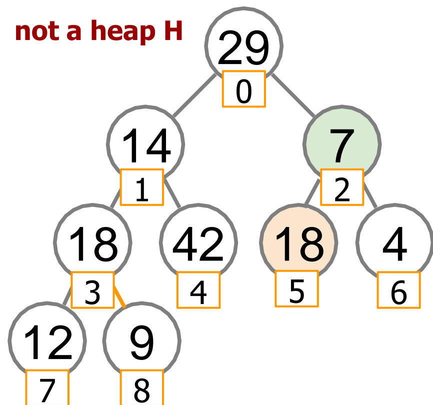
How to Heapify an array



➔ All the nodes H[3...8] are now fixed

29	14	7	18	42	18	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

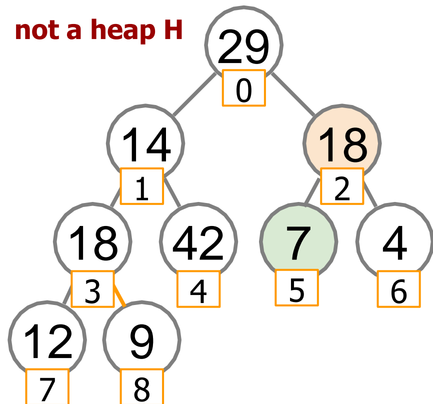


➔ the next node we need to fix is at position 2 of the array

sift_down(2)

29	14	7	18	42	18	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

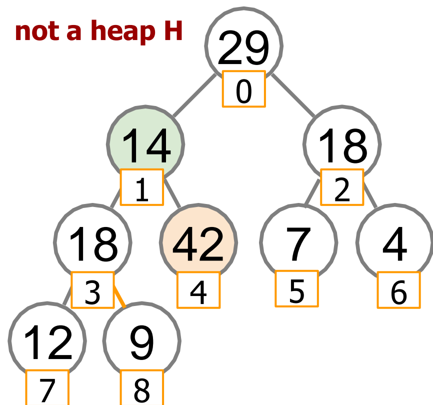


➔ the next node we need to fix is at position 2 of the array

sift_down(2)

29	14	18	18	42	7	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

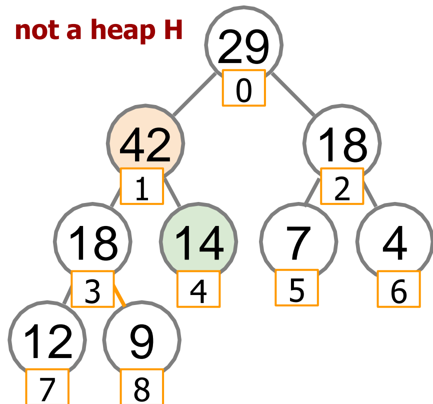


➔ the next node we need to fix is at position 1 of the array

sift_down(1)

29	14	18	18	42	7	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

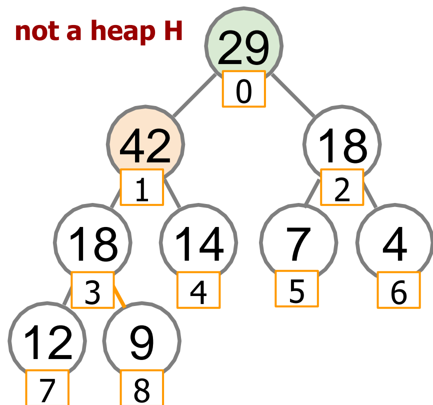


➔ the next node we need to fix is at position 1 of the array

sift_down(1)

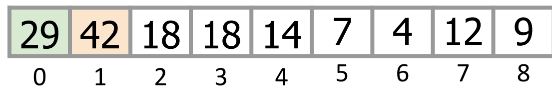
29	42	18	18	14	7	4	12	9
0	1	2	3	4	5	6	7	8

How to Heapify an array

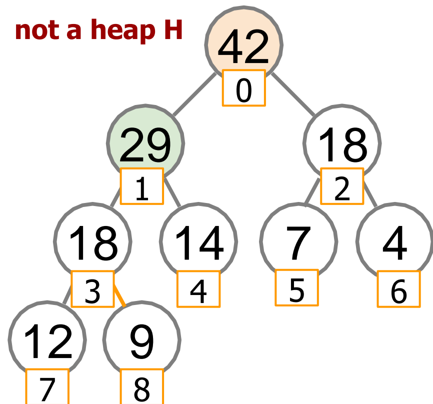


➔ Finally, we fix the root at position 0

sift_down(0)



How to Heapify an array



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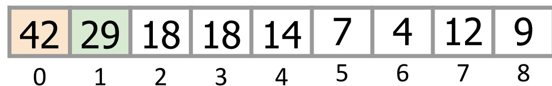
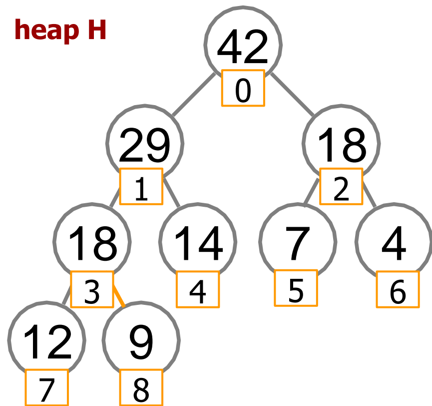
sift_down(0)



How to Heapify an array



42	29	18	18	14	7	4	12	9
0	1	2	3	4	5	6	7	8

- ➔ We rearranged the elements of the input array such that it is now a heap
- ➔ Next, we can use *extract max* inside the array itself to sort it in-place

Turning Array into a Heap

Heapify (array A of size n)

$last \leftarrow n - 1$

for i from $\lfloor (last - 1)/2 \rfloor$ down to 0:

    sift_down ($A[i]$)

In-place Heap Sort

HeapSort (array A of size n)

Heapify (A)
repeat $(n - 1)$ times:
    swap $A[0]$ and $A[size\text{-}1]$
    $size \leftarrow size - 1$
    sift_down(A[0])

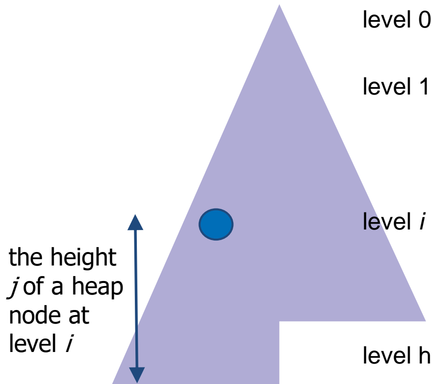
Running time: $O(n \log n)$

No additional space (in-place)

Run-time of Heapify

- The running time of *Heapify* is $O(n \log n)$ since we call *sift_down* for $O(n)$ nodes

The height of nodes at level *i*



Definition

If we count levels of the heap from top to bottom, then the ***height*** of a heap node at level i is defined to be $j = h - i$, where h is the total height of the heap

When we are repairing the heap, for each node at level i we need to swap at most j values

Run-time of *Heapify*

- The running time of *Heapify* is $O(n \log n)$ since we call *sift_down* for $O(n)$ nodes
- If a node is a leaf then we do not call *sift_down* on it
- If a node is close to the leaves, then sifting it down does not take $\log n$
- We have many such nodes!
- Is our estimate of the running time of *Heapify* too pessimistic?

Run-time of Heapify

level	# nodes	node height
h-h	2^{h-h}	h
	...	...
h-2	2^{h-2}	2
h-1	$\leq 2^{h-1}$	1
h-0	$\leq 2^{h-0}$	0

Total work:

$$\sum_{j=0}^{h} j * 2^{h-j} = 2^h \sum_{j=0}^{h} j * \frac{1}{2^j}$$

, where *j* represents the height of the nodes at each of 0…*h* tree levels

Evaluating the total work

Consider the sum of infinite convergent geometric series with x<1 and $a_0=1$

$$\sum_{j=0}^{\infty} x^j = \frac{1}{1-x} \qquad (1)$$

Take a derivative with respect to x of both parts of equality (1):

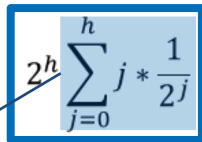
$$\sum_{j=0}^{\infty} j * x^{j-1} = \frac{1}{(1-x)^2} \qquad (2)$$

Multiply both sides of (2) by x:

$$\sum_{j=0}^{\infty} j * x^{j} = \frac{x}{(1-x)^2} \qquad (3)$$

Substitute x=½

$$\sum_{j=0}^{\infty} j * \frac{1}{2^j} = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$$



$$2^h \sum_{j=0}^{h} j * \frac{1}{2^j}$$

this sum evaluates to at most 2 even for infinite series!

This expression evaluates to O(n)

$$2^h \sum_{j=0}^{h} j * \frac{1}{2^j} \leq 2^h * 2 = O(2^h) = O(2^{\log n}) = O(n)$$

The running time of Heapify is ***O*(*n*)**

To convert an arbitrary array into a heap takes linear time and no additional space!

Top-k Problem

Input: An array A of size n, an integer $1 \le k \le n$.

Output: ***k largest*** elements of A (top-k).

Can be solved in time: $O(n) + O(k \log n)$

`Top-k` (array A of size n, k)

```
Heapify (A)
for i from 1 to k:
    extract_max()
```

Binary **Min**-Heap

Definition

Binary **min**-heap is a binary tree where the value of each node is **at most** the values of its children.

Can be implemented similarly to max-heap

Summary: Priority Queues

- *Priority queue* supports two main operations: *insert* and *pop* (extract the element with max priority)
- In an array/list implementation one operation is very fast ($O(1)$) but the other one is very slow ($O(n)$)
- *Binary heap* gives an implementation where both operations take $O(\log n)$ time
- The binary heap implementation is also space-efficient, because it is stored in an array